

# Anisotropy of the Cosmic Background Radiation implies the Violation of the Strong Energy Condition in Bianchi type I Universe

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## Abstract

We consider the horizon problem in a homogeneous but anisotropic universe (Bianchi type I). We show that the problem cannot be solved if (1) the matter obeys the strong energy condition with the positive energy density and (2) the Einstein equations hold. The strong energy condition is violated during cosmological inflation.

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## I. INTRODUCTION

The discovery of the cosmic microwave background (CMB) [1] verified the hot big bang cosmology. The high degree of its isotropy [2], however, gave rise to the horizon problem: Why could causally disconnected regions be isotropized? The inflationary universe scenario [3] may solve the problem because inflation made it possible for the universe to expand enormously up to the presently observable scale in a very short time. However inflation is the sufficient condition even if the cosmic no hair conjecture [4] is proved. Here, a problem again arises: Is inflation the unique solution to the horizon problem? What is the general requirement for the solution of the horizon problem?

Recently, Liddle showed that in FRW universe the horizon problem cannot be solved without violating the strong energy condition if gravity can be treated classically [5]. Actually the strong energy condition is violated during inflation. The generalization of his result to a more general inhomogeneous and anisotropic universe is urgent. The motivations are two folds. (1) The universe around the Planck epoch is expected to be highly inhomogeneous and anisotropic. (2) Even from the Planck epoch afterwards, the universe may be highly inhomogeneous, because it may experience many stages of phase transition, such as GUT, electroweak, quark-hadron, etc. Since the particle horizon from the Planck time to the time of nucleosynthesis is essential to Liddle's argument, we need to generalize his argument to an inhomogeneous and anisotropic universe. In short we are concerned about the "structural stability" of Liddle's argument. That is, is his result specific to FRW universe, or does it hold quite generally?

Maartens, Ellis, and Stoeger [6] recently showed that if the residual dipole of the cosmic microwave background anisotropy vanishes to first order of perturbations and if the quadrupole and the octapole are spatially homogeneous to first order, then the spacetime is locally Bianchi I to first order since the decoupling to the present day. Therefore as a first step towards the general case, we shall consider a homogeneous but anisotropic universe (Bianchi type I) in this Letter.

In Sec.2, we compute the comoving Hubble distance in a homogeneous but anisotropic universe. We find that the horizon problem again cannot be solved without violating the strong energy condition. In Sec.3, we give an argument generalizing to an inhomogeneous universe. Sec.4 is devoted to summary.

## II. HORIZON PROBLEM IN HOMOGENEOUS BUT ANISOTROPIC UNIVERSE

The Bianchi I universe is described by the following metric

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2. \quad (2.1)$$

We normalize each scale factor such that  $X(t) = Y(t) = Z(t) = 1$  at the present time ( $t_0$ ). We define the averaged scale factor by  $R^3(t) = X(t)Y(t)Z(t)$ . We assume that the energy-momentum tensor of the matter obeys

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (2.2)$$

From the Einstein equations we have

$$\frac{\dot{X}}{X} = \frac{\dot{R}}{R} + \frac{\sigma_x}{R^3}, \quad (2.3)$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{R}}{R} + \frac{\sigma_y}{R^3}, \quad (2.4)$$

$$\frac{\dot{Z}}{Z} = \frac{\dot{R}}{R} + \frac{\sigma_z}{R^3}, \quad (2.5)$$

$$\sigma_x + \sigma_y + \sigma_z = 0, \quad (2.6)$$

$$H^2 \equiv \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\sigma_0^2}{6R^6}, \quad (2.7)$$

$$\sigma_0^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2, \quad (2.8)$$

where dots denote the time derivative and  $\sigma_i$  is a constant representing the present shear in  $i$ th-direction. The Bianchi identity reads

$$(\rho R^3)^\cdot + p(R^3)^\cdot = 0. \quad (2.9)$$

The anisotropy of cosmic microwave background seen by COBE sets a limit on  $\sigma_0$  [7]

$$\frac{\sigma_0}{3H_0} \leq 6.9 \times 10^{-10}. \quad (2.10)$$

Without loss of generality we can impose the condition

$$\sigma_x \geq \sigma_y, \quad \sigma_x \geq \sigma_z. \quad (2.11)$$

With the help of Eq. (2.3)-(2.5) these conditions imply

$$X(t) \leq Y(t), \quad X(t) \leq Z(t) \quad (t \leq t_0), \quad (2.12)$$

since we have normalized each scale factor such that  $X(t_0) = Y(t_0) = Z(t_0) = 1$ . Thus, for  $t \leq t_0$ ,

$$(dx^2 + dy^2 + dz^2)^{1/2} \leq \frac{dt}{X(t)} \quad (2.13)$$

along any null lines. This means that the comoving distance  $d_{comm}$  along an arbitrary null line from  $t = t_1$  to  $t = t_2$  ( $t_{pl} \leq t_1 < t_2 \leq t_0$ ), which can be regarded as the communication distance in the terminology of Liddle, is bounded above as

$$d_{comm}(t_1, t_2) \equiv \int_{\text{along null}} \sqrt{dx^2 + dy^2 + dz^2} \leq \int_{t_1}^{t_2} \frac{dt}{X(t)}. \quad (2.14)$$

Now we have only to consider  $X(t)$  among  $X(t)$ ,  $Y(t)$  and  $Z(t)$  to show that  $d_{comm}(t_{pl}, t) \ll 1/H_0$  below. First let us define an effective density  $\rho_x$  and an effective pressure  $p_x$  by

$$\left( \frac{\dot{X}}{X} \right)^2 = \frac{8\pi G}{3} \rho_x, \quad (2.15)$$

$$\dot{\rho}_x = -3(\rho_x + p_x) \frac{\dot{X}}{X}. \quad (2.16)$$

Using  $x - x$  component of the evolution equation of the extrinsic curvatures, we have

$$\rho_x = \rho + \frac{\sigma_0^2}{16\pi GR^6} + \frac{3\sigma_x^2}{8\pi GR^6} + \frac{3\sigma_x \dot{R}}{4\pi GR^4}, \quad (2.17)$$

$$p_x = p + \frac{\sigma_0^2}{16\pi GR^6} - \frac{3\sigma_x^2}{8\pi GR^6}, \quad (2.18)$$

and

$$\rho_x + 3p_x = (\rho + 3p) + \frac{1}{4\pi GR^6} (\sigma_0^2 - 3\sigma_x^2 + 3\sigma_x R^2 \dot{R}). \quad (2.19)$$

Now let us assume  $\rho \geq 0$ . From Eq. (2.11), (2.6) and (2.8),  $\sigma_x$  is bounded from below and above as  $0 \leq \sigma_x \leq \sqrt{6}|\sigma_0|/3$ . Next using the positivity of the density,  $R^2 \dot{R} \geq |\sigma_0|/\sqrt{6}$  is derived from Eq. (2.7). Then we can prove that  $\rho_x + 3p_x$  is greater than  $\rho + 3p$  as

$$(\rho_x + 3p_x) - (\rho + 3p) \geq \frac{1}{4\pi GR^6} \left( \sigma_0^2 - 3\sigma_x^2 + \frac{\sqrt{6}}{2} \sigma_x |\sigma_0| \right) \quad (2.20)$$

$$= \frac{1}{4\pi GR^6} \left( |\sigma_0| - \frac{\sqrt{6}}{2} \sigma_x \right) (|\sigma_0| + \sqrt{6} \sigma_x) \quad (2.21)$$

$$\geq 0. \quad (2.22)$$

We thus finally have the strong energy condition for  $\rho_x$  and  $p_x$  as

$$\rho_x + 3p_x \geq \rho + 3p \geq 0. \quad (2.23)$$

provided that the original version of the strong energy condition holds ( $\rho + 3p \geq 0$ ).

Now under the above strong energy condition we shall prove the relation as

$$d_{comm}(t_{pl}, t) \leq \int_{t_{pl}}^t \frac{dt'}{X(t')} << \frac{1}{H_0}, \quad (2.24)$$

where  $t$  may be taken to be any time between the Planck time and the decoupling time. We take the standpoint that the matter content of the Universe is well understood after the big bang nucleosynthesis. In order to evaluate the integral in Eq.(2.24), we divide the time range into two epochs: (1) from the Planck time  $t_{pl}$  to the time of nucleosynthesis  $t_{nuc}$  which is defined by the time when the neutron-to-proton ratio is frozen out; (2) from  $t_{nuc}$  to the time of the decoupling of the microwave background  $t_{dec}$ . In epoch (1), we assume that matter obeys the energy condition such that

$$\rho_x + 3p_x \geq 0, \quad (2.25)$$

which has been derived under the strong energy condition with positive energy density. In epoch (2), the universe is dominated by the ordinary dust matter and radiation.

### A. from $t_{pl}$ to $t_{nuc}$

From Eqs.(2.15) and (2.16) we have

$$\frac{d \ln X}{d \ln \rho_x} = -\frac{1}{3} \frac{\rho_x}{\rho_x + p_x}, \quad (2.26)$$

which clearly shows that  $X$  decrease most rapidly when the pressure  $p_x$  is the lowest  $p_x = -\rho_x/3$ . The integral of Eq.(2.24) can be rewritten as

$$\begin{aligned} \int_{t_{pl}}^{t_{nuc}} \frac{dt'}{X(t')} &= - \int_{(\rho_x)_{pl}}^{(\rho_x)_{nuc}} \frac{d\rho_x}{3H_x X(\rho_x)(\rho_x + p_x)} \\ &= -\frac{1}{\sqrt{24\pi G}} \int_{(\rho_x)_{pl}}^{(\rho_x)_{nuc}} \frac{d\rho_x}{X(\rho_x + p_x)\sqrt{\rho_x}}, \end{aligned} \quad (2.27)$$

where

$$H_x \equiv \frac{\dot{X}}{X} \quad (2.28)$$

The integral is maximized by the lowest pressure [5] and we have

$$d_{comm}(t_{pl}, t_{nuc}) \leq \frac{1}{2X_{nuc}(H_x)_{nuc}} \ln \frac{(\rho_x)_{pl}}{(\rho_x)_{nuc}}. \quad (2.29)$$

From the definition of  $\rho_x$  in Eq.(2.17),  $(\rho_x)_{pl}$  is maximized when  $\sigma_0, \sigma_x$  and  $\dot{R}$  are maximized and  $R_{pl}$  is minimized.  $R_{pl}$  is minimized by the lowest possible pressure  $p_x = -\rho_x/3$ . Since  $t_{pl} \sim 10^{-43}$ sec and  $t_{nuc} \sim 1$ sec, we have

$$\begin{aligned} (\rho_x)_{pl} &= \rho_{pl} + \frac{\sigma_0^2}{16\pi G R^6} + \frac{3\sigma_x^2}{8\pi G R^6} + \frac{3\sigma_x \dot{R}}{4\pi G R^4} \\ &\lesssim \rho_{pl} + \frac{9}{16\pi G} \frac{\sigma_0^2}{(R_{pl})^6} \\ &\lesssim \rho_{pl} + \frac{9}{16\pi G} \frac{(10^{-9} \times H_0)^2}{(10^{-32})^6}, \end{aligned} \quad (2.30)$$

where  $\rho_{pl} \sim (10^{19}\text{Gev})^4$  is the Planck energy. Thus we can estimate the right hand side of Eq.(2.29) as

$$\ln \frac{(\rho_x)_{pl}}{(\rho_x)_{nuc}} < 620, \quad (2.31)$$

where  $(\rho_x)_{nuc} \sim (10^{-3}\text{Gev})^4$ . Since  $H_{nuc}^{-1}/R_{nuc} \simeq 10^{-4}\text{Mpc}$ , we find that at most

$$d_{comm}(t_{pl}, t_{nuc}) < 1 \times 10^{-5}/H_0. \quad (2.32)$$

### B. from $t_{nuc}$ to $t_{dec}$

Since the shear is negligible in this epoch, the analysis is completely the same as Liddle's [5]. Here we repeat his analysis for completeness. Between nucleosynthesis and decoupling, the universe is either in the stage of radiation-dominant or matter-dominant and the distance is bounded above by assuming matter is dominated throughout

$$d_{comm}(t_{nuc}, t_{dec}) \leq \frac{2}{R_{dec}H_{dec}}. \quad (2.33)$$

Since  $H_{dec}^{-1}/R_{dec} \simeq 100\text{Mpc}$ , we find

$$d_{comm}(t_{nuc}, t_{dec}) \leq 6.6 \times 10^{-2}/H_0. \quad (2.34)$$

To conclude, we cannot have  $d_{comm}(t_{pl}, t_{dec}) > 1/H_0$  in a homogeneous but anisotropic universe.

### III. GENERALIZING TO INHOMOGENEOUS UNIVERSES

In the previous section we have shown that the horizon problem is not solved under the strong energy condition even in a homogeneous but anisotropic universe considering the current limit on the anisotropy of the universe set by CMB. Therefore the strong energy condition should have been violated in the early universe. However a natural question arises: How can we generalize the above result to an inhomogeneous universe? Here we show that if the concept of the averaged scale factor makes sense, we can prove the “no-go theorem”. More precisely we show the following:

#### **Theorem**

*If we assume that*

- 1) *the universe can be foliated by geodesic slicing,*
- 2) *matter satisfies the strong energy condition  $(T_{ab} - \frac{1}{2}g_{ab}T)n^an^b \geq 0$ ,*
- 3) *the spatial curvature is everywhere not positive  ${}^3R \leq 0$ ,*
- 4)  *${}^3R$  and  $\sigma_{ab}\sigma^{ab}$  take the value of order the Planck scale at  $t_{pl}$ ,*

*then the inequality  $\int dt/R(t) < 1/H_0$  follows for the averaged scale factor  $R$  defined below.*

We take the geodesic slice, then the following two equations are necessary in our argument:

$$\begin{aligned} \frac{2}{3}K^2 &= -{}^3R + \sigma_{ab}\sigma^{ab} + 16\pi GT_{ab}n^an^b, \\ \dot{K} &= -\frac{1}{3}K^2 - \sigma_{ab}\sigma^{ab} - 8\pi G(T_{ab} - \frac{1}{2}g_{ab}T)n^an^b, \end{aligned} \quad (3.1)$$

where  $n^a$  is the unit normal to the spacelike hypersurface,  $K$  is the trace of the extrinsic curvature  $K_{ab}$ , and the shear tensor  $\sigma_{ab}$  is defined by

$$K_{ab} = \frac{1}{2}\dot{h}_{ab} = \frac{1}{3}K h_{ab} + \sigma_{ab}. \quad (3.2)$$

$h_{ab}$  is the spatial metric  $h_{ab} = g_{ab} + n_a n_b$ . We define the effective Hubble parameter (“volume expansion rate” [8])  $H$  and the effective scale factor  $R$  by

$$\begin{aligned} H &= \frac{1}{3}K \\ \frac{\dot{R}}{R} &= H. \end{aligned} \quad (3.3)$$

Similarly, we can define an effective density  $\tilde{\rho}$  and an effective pressure  $\tilde{p}$  by

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\tilde{\rho}, \\ \dot{\tilde{\rho}} &= -3H(\tilde{\rho} + \tilde{p}). \end{aligned} \quad (3.4)$$

$\tilde{\rho}$  and  $\tilde{p}$  are written as

$$\tilde{\rho} = T_{ab}n^a n^b + \frac{1}{16\pi G}(\sigma_{ab}\sigma^{ab} - {}^3R), \quad (3.5)$$

$$\tilde{p} = \frac{1}{3}T_{ab}h^{ab} + \frac{1}{48\pi G}({}^3R + 3\sigma_{ab}\sigma^{ab}). \quad (3.6)$$

We first show that  $\tilde{\rho}$  is positive as

$$\tilde{\rho} = T_{ab}n^a n^b + \frac{1}{16\pi G}(\sigma_{ab}\sigma^{ab} - {}^3R) \geq \rho > 0. \quad (3.7)$$

We can also show that the strong energy condition for  $\tilde{\rho}$  and  $\tilde{p}$  as

$$\tilde{\rho} + 3\tilde{p} = \rho + T_{ab}h^{ab} + \frac{1}{4\pi G}\sigma_{ab}\sigma^{ab} \geq 0, \quad (3.8)$$

if the strong energy condition  $((T_{ab} - \frac{1}{2}g_{ab}T)n^a n^b \geq 0)$  is satisfied. The proof of the theorem may follow by replacing  $X, \rho_x$  and  $p_x$  in the previous section with  $R, \tilde{\rho}$  and  $\tilde{p}$ , respectively.

#### IV. SUMMARY

We have shown that in a homogeneous but anisotropic (Bianchi I) universe the horizon problem cannot be solved if (1) matter satisfies the strong energy condition and (2) the Einstein equations hold. Changing the gravity theory would not change the result as shown by Liddle [5]. It would be very interesting to note that the anisotropy of CMB alone may imply that anomalous phenomena must have happened in the very early stage of the universe: the violation of the strong energy condition.

It should be noted that the interpretation of the origin of fluctuations of cosmic microwave background is not yet conclusive. Topological defects models can generate large angle cosmic microwave background fluctuations. Such models, however, do not generate perturbations well above the Hubble radius. Future observations will distinguish inflation from defect models [9].

It is also noted that our analysis as well as Liddle's is within the realm of classical theory. We have to keep in mind that the inflation may not be the only solution to the horizon problem. In fact, the correlation beyond the horizon does exist in any quantum field theory [10]. The existence of correlations beyond the horizon might have played an important role in the early universe. What we have shown here is that in a homogeneous but anisotropic universe the horizon problem may be solved either by the causal processes during a period of inflation or the acausal processes of quantum gravity.

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